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Best Constants in Preservation Inequalities Concerning the First Modulus and Lipschitz Classes for Bernstein-Type Operators*

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We consider families $(L_t, t \in T)$ of positive linear operators such that each L_t is representable in terms of a stochastic process starting at the origin and having non-decreasing paths and integrable stationary increments. For these families, we give probabilistic characterizations of the best possible constants both in preservation inequalities concerning the first modulus and in preservation of Lipschitz classes of first order. As an application, we compute such constants for the Bernstein, Szász, Gamma, Baskakov, and Beta operators. © 1998 Academic Press

1. INTRODUCTION AND MAIN RESULT

Many families $L := (L_t, t \in T)$ of positive linear operators usually considered in the literature on approximation theory allow for a probabilistic representation of the form (cf. [1, 2])

$$L_t f(x) = Ef(Z_t(x)), \qquad x \in I, \quad t \in T, \tag{1}$$

where $I = [0, 1]$ or $I = [0, \infty)$, $T = \{1, 2, ...\}$ or $T = (0, \infty)$, E denotes mathematical expectation, $(Z_t(x), x \in I, t \in T)$ is a double-indexed stochastic process of integrable random variables taking values in I, and f is any real measurable function on I for which the right-hand side in (1) is well defined.

The aim of this note is to obtain the best possible constants both in preservation inequalities referring to the first modulus of continuity and in preservation of Lipschitz classes of first order, for families L of operators

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having the form (1) and such that, for each $t \in T$, the stochastic process $(Z_t(x), x \in I)$ satisfies the following properties:

(A) *Nondecreasing paths.* For all $x, y \in I$ with $x \leqslant y$, we have $Z_t(x) \leqslant Z_t(y)$ a.s.

(B) $Z_t(0) = 0$ a.s.

(C) *Stationary increments.* For all $x, y \in I$ with $x \leqslant y$, the random variables $Z_t(y) - Z_t(x)$ and $Z_t(y-x) - Z_t(0)$ are identically distributed.

As it is shown in [2], the most usual positive linear operators fulfill condition (A). However, some well-known operators do not satisfy (B) or (C) (see Section 3).

Under the preceding assumptions, we show that such best constants can be characterized as expectations of appropriate functions of the stochastic process under consideration (Theorem 1). Thanks to this characterization and the probabilistic structure of the process involved, we are able to provide exact values of these constants for certain important families of operators (see Section 2).

More precisely, recall that the usual first modulus of continuity of a real function f defined on I is given by

$$\omega(f; \delta) := \sup \{ |f(x+h) - f(x)| : x, x+h \in I, 0 < h \leqslant \delta \}, \qquad \delta \in I^*,$$

where $I^* := I \backslash \{0\}$. Denote by $M(I)$ the set of all real measurable functions f defined on I such that $\omega(f; \delta) < \infty$, $\delta \in I^*$. Also, recall that, for any $f \in M(I)$ and $\delta \in I^*$, we have

$$\omega(f; x) \leqslant \omega(f; \delta) \left\lceil \frac{x}{\delta} \right\rceil, \qquad x \in I, \tag{2}$$

where $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Finally, denote by $\operatorname{Lip}(A, \alpha)$ the Lipschitz class of first order with constant $A > 0$ and exponent $\alpha \in (0, 1]$, i.e.,

$$\operatorname{Lip}(A, \alpha) := \{ f \in M(I) : \omega(f; \delta) \leqslant A\delta^\alpha, \delta \in I^* \}.$$

Observe that if $f \in M(I)$, $L_t f(x)$ is well defined for all $x \in I$ and $t \in T$. Actually, assumptions (A) and (B), together with (2), give us

$$\begin{aligned} L_t |f|(x) &\leqslant |f(0)| + E\omega(|f|; Z_t(x)) \\ &\leqslant |f(0)| + \omega(|f|; 1)\, E \lceil Z_t(x) \rceil < \infty. \end{aligned}$$

As far as the first modulus is concerned, we shall be interested in the constants

$$C_t(\delta) := \sup_{f \in M(I)} \frac{\omega(L_t f; \delta)}{\omega(f; \delta)}, \qquad \delta \in I^*, \quad t \in T, \tag{3}$$

$$C_t := \sup_{\delta \in I^*} C_t(\delta), \qquad t \in T, \qquad C := \sup_{t \in T} C_t. \tag{4}$$

With respect to the Lipschitz classes, we shall consider, for any $\alpha \in (0, 1]$, the constants

$$K_t(\alpha) := \sup_{\delta \in I^*} \sup_{f \in \mathrm{Lip}(1, \alpha)} \frac{\omega(L_t f; \delta)}{\delta^\alpha}, \qquad t \in T, \tag{5}$$

$$K(\alpha) := \sup_{t \in T} K_t(\alpha). \tag{6}$$

As far as we know, preservation inequalities concerning the first modulus of continuity for families L of discrete operators were first obtained by Kratz and Stadtmüller [11]. Better estimates for more general families can be found in [1]. On the other hand, Lindvall [12] and Brown, Elliott and Paget [6], among others, have shown that the Bernstein polynomials preserve Lipschitz constants. In a more general setting, a probabilistic approach to this problem is given in [1, 10].

The question, however, of finding the best possible constants has only been considered with regard to the first modulus of continuity and for particular families of operators. For instance, Anastassiou, Cottin, and Gonska [5] provide the best absolute constant in the case of Bernstein polynomials on the standard m-simplex, while for the classical Szász operator, we refer to [4].

With the notations above, our main result is stated as follows.

THEOREM 1. *Let $(L_t, t \in T)$ be a family of positive linear operators having the form* (1) *and satisfying assumptions* (A)–(C). *Then, for each $t \in T$, we have*

(a) $C_t(\delta) = E\lceil Z_t(\delta)/\delta \rceil = \sum_{k=0}^{\infty} P(Z_t(\delta) > k\delta)$, $\delta \in I^*$.

(b) $K_t(\alpha) = \sup_{\delta \in I^*} E(Z_t(\delta)/\delta)^\alpha$, $\alpha \in (0, 1]$.

Proof. Let $t \in T$ be fixed. Let $f \in M(I)$, $\delta \in I^*$, and $x, x+h \in I$ with $0 < h \leqslant \delta$. From (A)–(C), we have

$$\begin{aligned} |L_t f(x+h) - L_t f(x)| &\leqslant E\,|f(Z_t(x+h)) - f(Z_t(x))| \\ &\leqslant E\omega(f; Z_t(x+h) - Z_t(x)) \\ &\leqslant E\omega(f; Z_t(x+\delta) - Z_t(x)) \\ &= E\omega(f; Z_t(\delta)), \end{aligned}$$

implying that

$$\omega(L_t f;\delta) \leqslant E\omega(f; Z_t(\delta)), \qquad \delta \in I. \tag{7}$$

Denote by $S(I)$ the set of all nondecreasing subadditive functions $f \in M(I)$ such that $f(0)=0$. Observe that if $f \in S(I)$, then $\omega(f;x)=f(x)$, $x \in I$. On the other hand, $L_t(S(I)) \subseteq S(I)$. In fact, if $f \in S(I)$, $L_t f$ is nondecreasing and satisfies $L_t f(0)=0$, by assumptions (A) and (B), respectively. The subadditivity of $L_t f$ follows from (A)–(C), since

$$\begin{aligned} L_t f(x+y) &= Ef(Z_t(x+y)) \\ &\leqslant Ef(Z_t(x)) + Ef(Z_t(x+y) - Z_t(x)) \\ &= L_t f(x) + L_t f(y), \qquad x, y \in I. \end{aligned}$$

We therefore conclude that for any $f \in S(I)$

$$\omega(f;\delta) = f(\delta), \qquad \omega(L_t f;\delta) = L_t f(\delta), \qquad \delta \in I. \tag{8}$$

(a) By (7) and (2), we have

$$C_t(\delta) \leqslant E\left\lceil \frac{Z_t(\delta)}{\delta} \right\rceil.$$

To prove the converse inequality, define $g_\delta(x) := \lceil x/\delta \rceil$, $x \in I$. Then $g_\delta \in S(I)$ and satisfies $g_\delta(\delta)=1$. Hence, we have from (8)

$$\omega(L_t g_\delta;\delta) = L_t g_\delta(\delta) = \omega(g_\delta;\delta)\, E\left\lceil \frac{Z_t(\delta)}{\delta} \right\rceil.$$

The conclusion follows.

(b) Fix $\alpha \in (0,1]$. Applying (7) to any function $f \in \mathrm{Lip}(1,\alpha)$, we obtain

$$K_t(\alpha) \leqslant \sup_{\delta \in I^*} E(Z_t(\delta)/\delta)^\alpha.$$

On the other hand, consider the function $f_1(x) = x^\alpha$, $x \in I$. Since $f_1 \in \mathrm{Lip}(1,\alpha) \cap S(I)$, we have from (8)

$$\omega(L_t f_1;\delta) = L_t f_1(\delta) = \delta^\alpha E(Z_t(\delta)/\delta)^\alpha, \qquad \delta \in I^*.$$

This completes the proof of Theorem 1.

Remark 1. Denote by $C(I)$ the set of all real continuous functions defined on I. We claim that

$$C_t(\delta) = C_t^*(\delta) := \sup_{f \in M(I) \cap C(I)} \frac{\omega(L_t f; \delta)}{\omega(f; \delta)}, \qquad \delta \in I^*, \quad t \in T.$$

Indeed, for any $0 < \varepsilon < \delta$, we consider the function $g_{\delta, \varepsilon} \in M(I) \cap C(I)$ given by

$$g_{\delta, \varepsilon}(x) = \sum_{k=0}^{\infty} \left(\frac{x - k\delta}{\varepsilon} - 1 \right) 1_{(k\delta, k\delta + \varepsilon)}(x) + g_\delta(x), \qquad x \in I,$$

where, as before, $g_\delta(x) = \lceil x/\delta \rceil$, $x \in I$. Observe that $\lim_{\varepsilon \to 0} g_{\delta, \varepsilon}(x) = g_\delta(x)$, $x \in I$, and $\omega(g_{\delta, \varepsilon}; \delta) = 1$, $0 < \varepsilon < \delta$. Thus, using Theorem 1(a) and dominated convergence, we obtain

$$\begin{aligned} C_t(\delta) = L_t g_\delta(\delta) &= \lim_{\varepsilon \to 0} \, (L_t g_{\delta, \varepsilon}(\delta) - L_t g_{\delta, \varepsilon}(0)) \\ &\leqslant \limsup_{\varepsilon \to 0} \omega(L_t g_{\delta, \varepsilon}; \delta) \leqslant C_t^*(\delta), \end{aligned}$$

which shows the claim.

Remark 2. It is readily seen from definition (3) that the function $C_t(\delta)$, $\delta \in I^*$, is subadditive for any $t \in T$. On the other hand, we have from Theorem 1(a) the upper bound

$$C_t(\delta) \leqslant P(Z_t(\delta) > 0) + E\left(\frac{Z_t(\delta)}{\delta} \right), \qquad \delta \in I^*, \quad t \in T.$$

In particular, $C_t(\delta) \leqslant 2$, whenever $EZ_t(\delta) = \delta$.

2. EXAMPLES

In this section, we consider classical families of Bernstein-type operators which allow for a probabilistic representation of the form (1) and satisfy assumptions (A)–(C). All the following representations were already given in [2]. Exact values of the constants (3)–(6) are obtained by using Theorem 1 and the stochastic properties of the process involved in each case. In all the following examples, the operators under consideration are centered, that is, $EZ_t(\delta) = \delta$, $\delta \in I$, $t \in T$.

(A) *Bernstein Operator.* The Bernstein polynomials of a real function f on $[0, 1]$ can be represented as

$$B_n f(x) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}$$
$$= Ef\left(\frac{S_n(x)}{n}\right), \qquad x \in [0, 1], \quad n = 1, 2, ...,$$

where

$$S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(X_k), \qquad x \in [0, 1], \quad n = 1, 2, ...$$

and $(X_k)_{k \geqslant 1}$ is a sequence of independent and on the interval $[0, 1]$ uniformly distributed random variables.

By Theorem 1(a) and [8, p. 59], we have for any $\delta \in (0, 1]$ and $n = 1, 2, ...$

$$C_n(\delta) = \sum_{k=0}^{[1/\delta]} P(S_n(\delta) > kn\delta) = \int_0^{\delta} \sum_{k=0}^{[1/\delta]} \frac{x^{[kn\delta]}(1-x)^{n-[kn\delta]-1}}{\beta([kn\delta]+1, n-[kn\delta])} \, dx, \quad (9)$$

where $[x]$ stands for the integral part of x, $\beta(\cdot, \cdot)$ is the beta function, and it is understood that $\beta(\cdot, 0) = \infty$.

For any $n = 1, 2, ...$ and $\delta \in (1 - 1/n, 1)$, it follows from (9) and Remark 2 that $1 - (1-\delta)^n + \delta^n \leqslant C_n(\delta) \leqslant 2$, showing that

$$C_n = C = 2, \qquad n = 1, 2, ...$$

This last result was also obtained in [5] using a different approach.

Finally, Theorem 1(b), together with Jensen's inequality, gives us

$$K_n(\alpha) = K(\alpha) = 1, \qquad \alpha \in (0, 1], \quad n = 1, 2, ...$$

We point out that the inequalities $K_n(\alpha) \leqslant 1$ and $K(\alpha) \leqslant 1$ have been shown in [5, Theorem 9] by applying a technique based on least concave majorants.

(B) *Szász–Mirakyan Operator.* For this operator, we have the representation

$$S_t f(x) := e^{-tx} \sum_{k=0}^{\infty} f(k/t) \frac{(tx)^k}{k!} = Ef\left(\frac{N_{tx}}{t}\right), \qquad x \geqslant 0, \quad t > 0,$$

where $(N_t)_{t \geqslant 0}$ is a standard Poisson process.

Using Theorem 1(a) and the well-known formula (cf. [8])

$$P(N_t \geqslant n) = \frac{1}{(n-1)!} \int_0^t e^{-x} x^{n-1} \, dx, \qquad t \geqslant 0, \quad n = 1, 2, ... \qquad (10)$$

we obtain for any $\delta > 0$ and $t > 0$,

$$C_t(\delta) = \sum_{k=0}^{\infty} P(N_{t\delta} > kt\delta) = \int_0^{t\delta} e^{-x} \sum_{k=0}^{\infty} \frac{x^{[kt\delta]}}{[kt\delta]!}\, dx,$$

which, thanks to [4, Lemma 2], implies that

$$C_t = C = 2 - \frac{1}{e}, \qquad t > 0.$$

Since $(N_t/t)_{t>0}$ converges to 1 almost surely, as $t \to \infty$, we have from Fatou's lemma and Jensen's inequality

$$1 \leqslant \liminf_{t \to \infty} E\left(\frac{N_t}{t}\right)^{\alpha} \leqslant \sup_{t>0} E\left(\frac{N_t}{t}\right)^{\alpha} \leqslant 1, \qquad \alpha \in (0, 1].$$

Therefore, Theorem 1(b) gives us

$$K_t(\alpha) = K(\alpha) = 1, \qquad \alpha \in (0, 1], \quad t > 0.$$

(C) *Gamma Operator.* A suitable probabilistic representation for this operator is

$$G_t f(x) := \frac{1}{\Gamma(t)} \int_0^{\infty} f\left(\frac{x\theta}{t}\right) \theta^{t-1} e^{-\theta}\, d\theta = Ef\left(\frac{xU_t}{t}\right), \qquad x \geqslant 0, \quad t > 0,$$

where $(U_t)_{t \geqslant 0}$ is a gamma process, i.e., a process starting at the origin, having stationary independent increments and such that, for each $t > 0$, U_t has the gamma density

$$d_t(\theta) := \frac{\theta^{t-1} e^{-\theta}}{\Gamma(t)}, \qquad \theta > 0, \tag{11}$$

$\Gamma(t)$ being the gamma function. Theorem 1(a) yields in this case for any $\delta > 0$ and $t > 0$

$$C_t(\delta) = C_t = E\left\lceil \frac{U_t}{t} \right\rceil = \frac{t^t}{\Gamma(t)} \int_0^{\infty} \lceil \theta \rceil\, \theta^{t-1} e^{-t\theta}\, d\theta. \tag{12}$$

Since $t^{-1} U_t$ converges to 0 in probability, as $t \to 0$, we have (cf. [7, p. 67])

$$\lim_{t \to 0} E\left(1 + \frac{U_t}{t} - \left\lceil \frac{U_t}{t} \right\rceil\right) = 0. \tag{13}$$

This, together with (12), immediately implies that $C = 2$.

By Theorem 1(b),

$$K_t(\alpha) = E\left(\frac{U_t}{t}\right)^{\alpha} = \frac{1}{t^{\alpha}}\,\frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad \alpha \in (0, 1], \quad t > 0.$$

Finally, the process $t^{-1}U_t$ converges to 1 almost surely, as $t \to \infty$, as it follows from the strong law of large numbers. Thus, as in example (B), we obtain

$$K(\alpha) = 1, \qquad \alpha \in (0, 1].$$

(D) *Baskakov Operator.* We give the following representation for the Baskakov operator

$$\begin{aligned} B_t^* f(x) &:= \sum_{k=0}^{\infty} f(k/t) \binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}} \\ &= Ef\left(\frac{N_{xU_t}}{t}\right), \qquad x \geqslant 0, \quad t > 0, \end{aligned}$$

where $(N_t)_{t \geqslant 0}$ is a standard Poisson process and $(U_t)_{t \geqslant 0}$ is a gamma process independent of $(N_t)_{t \geqslant 0}$.

Conditioning on U_t and using (10), we have for any $\delta > 0$ and $t > 0$

$$C_t(\delta) = \int_0^{\delta} \sum_{k=0}^{\infty} \frac{1}{\beta(t, [kt\delta]+1)}\, \frac{x^{[kt\delta]}}{(1+x)^{[kt\delta]+t+1}}\, dx.$$

In this case, we shall show that $C = 2$. In view of Remark 2, it will suffice to prove that $\lim_{t \to 0} \lim_{\delta \to \infty} C_t(\delta) = 2$. To this end, observe that

$$\lim_{\delta \to \infty} \frac{N_{\delta U_t}}{\delta t} = \frac{U_t}{t}, \qquad \text{a.s.}, \quad t > 0. \tag{14}$$

Since U_t is a continuous random variable, we have from Theorem 1(a), the dominated convergence theorem and (14) and (13)

$$\lim_{t \to 0} \lim_{\delta \to \infty} C_t(\delta) = \lim_{t \to 0} \lim_{\delta \to \infty} E\left|\frac{N_{\delta U_t}}{\delta t}\right| = \lim_{t \to 0} E\left|\frac{U_t}{t}\right| = 2.$$

On the other hand, applying successively Fatou's lemma, (14), and Jensen's inequality, we obtain

$$E\left(\frac{U_t}{t}\right)^{\alpha} \leqslant \liminf_{\delta\to\infty} E\left(\frac{N_{\delta U_t}}{\delta t}\right)^{\alpha} \leqslant \sup_{\delta>0} \int_0^{\infty} E\left(\frac{N_{\delta\theta}}{\delta t}\right)^{\alpha} d_t(\theta)\, d\theta$$

$$\leqslant \int_0^{\infty} \left(\frac{\theta}{t}\right)^{\alpha} d_t(\theta)\, d\theta = E\left(\frac{U_t}{t}\right)^{\alpha}, \qquad t>0,$$

where $d_t(\theta)$ is defined in (11). Consequently, from Theorem 1(b)

$$K_t(\alpha) = E\left(\frac{U_t}{t}\right)^{\alpha} = \frac{1}{t^{\alpha}} \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad \alpha\in(0,1], \quad t>0,$$

implying, as we have seen in example (C), that

$$K(\alpha)=1, \qquad \alpha\in(0,1].$$

(E) *Beta Operator.* We consider the beta operator B_t introduced by Mühlbach [13] (see also [3, 9]) for which we give the representation

$$B_t f(x) := \int_0^1 f(\theta) \frac{\theta^{tx-1}(1-\theta)^{t(1-x)-1}}{\beta(tx, t(1-x))}\, d\theta = Ef\left(\frac{U_{tx}}{U_t}\right), \qquad x\in[0,1], \quad t>0,$$

where $(U_t)_{t\geqslant 0}$ is a gamma process.

We can see that for any $\delta\in(0,1]$ and $t>0$,

$$C_t(\delta) = E\left\lceil \frac{U_{t\delta}}{\delta U_t} \right\rceil = \frac{\delta^{t\delta}}{\beta(t\delta, t(1-\delta))} \int_0^{1/\delta} \lceil\theta\rceil\, \theta^{t\delta-1}(1-\theta\delta)^{t(1-\delta)-1}\, d\theta. \qquad (15)$$

As in example (C), we have

$$\lim_{\delta\to 0} E\left(1+\frac{U_{t\delta}}{\delta U_t} - \left\lceil \frac{U_{t\delta}}{\delta U_t} \right\rceil\right) = 0.$$

This, together with (15) and Remark 2, gives

$$C_t = C = 2, \qquad t>0.$$

Finally, choosing $\delta=1$, Theorem 1(b) and Jensen's inequality immediately yield

$$K_t(\alpha) = K(\alpha) = 1, \qquad \alpha\in(0,1], \quad t>0.$$

3. CONCLUDING REMARKS

Apart from the preceding examples, there are other families of operators satisfying the assumptions made in Section 1, for which Theorem 1 is applicable, for instance, Müller gamma operators, inverse beta operators, and Stancu operators (cf. [2, Sect. 4]). On the other hand, Bernstein–Durrmeyer and Bernstein–Kantorovich operators, as well as their generalizations, do not satisfy condition (B). Bleimann–Butzer–Hahn operators and Meyer–König–Zeller operators, among others, do not satisfy condition (C). Notwithstanding, since all the aforementioned operators satisfy condition (A), we can give sharp upper bounds for all the corresponding constants (3)–(6), as it is shown in [1].

Under weaker assumptions than those considered in Section 1, it does not seem easy to provide exact values of all the constants (3)–(6), nor to obtain the constants $C_t(\delta)$ and $K_t(\alpha)$ in a closed form extending that established in Theorem 1. Instead, some partial results can be given for certain families of operators. We mention the following examples:

(a) *Convolution Operators.* These operators have the form

$$L_t f(x) = Ef(x+Z_t), \qquad -\infty < x < \infty, \quad t > 0, \tag{16}$$

where each Z_t is an integrable random variable whose distribution does not depend upon x. In particular, if Z_t has the normal distribution with zero mean and variance equal to $1/t$, we obtain the classical Weierstrass operator. It is readily seen from (16) that

$$C_t(\delta) = C_t = C = 1, \qquad \delta > 0, \quad t > 0.$$

(b) (*Modified*) *Meyer–König–Zeller Operator.* A probabilistic representation for this operator is

$$M_t f(x) := (1-x)^{t+1} \sum_{k=0}^{\infty} f\left(\frac{k}{k+t}\right) \binom{t+k}{k} x^k = Ef(Z_t(x)),$$

where

$$Z_t(x) := \frac{N_{q(x)\, U_{t+1}}}{N_{q(x)\, U_{t+1}} + t}, \qquad q(x) := \frac{x}{1-x}, \qquad x \in [0, 1), \quad t > 0, \tag{17}$$

$(N_t)_{t \geqslant 0}$ is a standard Poisson process, and $(U_t)_{t \geqslant 0}$ is a gamma process independent of $(N_t)_{t \geqslant 0}$. In this case, we shall show that $C = 2$. Actually, since condition (A) is satisfied and $EZ_t(x) = x$, $x \in [0, 1)$, $t > 0$, the inequality

$C \leqslant 2$ follows from [1, Corollary 2]. As for the converse inequality, observe that

$$C_t(\delta) \geqslant E\left\lceil \frac{Z_t(\delta)}{\delta} \right\rceil \geqslant P(Z_t(\delta) > 0) + P(Z_t(\delta) > \delta)$$

$$= 1 - (1-\delta)^{t+1} + P\left(\frac{N_{q(\delta)\, U_{t+1}}}{tq(\delta)} > 1 \right), \qquad \delta \in (0, 1), \quad t > 0,$$

as it follows from (17). Taking limits in the preceding inequalities as $\delta \to 1$ and using the strong law of large numbers for the Poisson process, we obtain

$$C \geqslant 1 + P\left(\frac{U_{t+1}}{t} > 1 \right), \qquad t > 0.$$

The conclusion follows by letting $t \to 0$.

(c) *Lipschitz Constants.* Let $(L_t, t \in T)$ be a family of operators of the form (1). Assume that condition (A) is satisfied and that $EZ_t(x) = x$, $x \in I$, $t \in T$, where I is allowed to be any subinterval of the real line.

Then, it is shown in [1, Corollary 1] that $K(\alpha) \leqslant 1$, $\alpha \in (0, 1]$. Under the following additional assumption

$$\lim_{t \to \infty} L_t f(x) = f(x), \qquad x \in I, \quad f \in \mathrm{Lip}(1, \alpha), \quad \alpha \in (0, 1], \tag{18}$$

we have

$$K(\alpha) = 1, \qquad \alpha \in (0, 1].$$

To see this, let $f_0 \in \mathrm{Lip}(1, \alpha)$ be such that $\omega(f_0; \delta) = \delta^\alpha$, $0 \leqslant \delta < l(I)$, where $l(I)$ denotes the length of I. For any x, $x + h \in I$ with $0 \leqslant h \leqslant \delta$, we have from (18)

$$|f_0(x+h) - f_0(x)| = \lim_{t \to \infty} |L_t f_0(x+h) - L_t f_0(x)| \leqslant K(\alpha)\, \delta^\alpha,$$

showing that $K(\alpha) \geqslant 1$. The conclusion follows.

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